

CONVERGENCE AND SUMMABILITY OF MULTIPLE FOURIER SERIES AND GENERALIZED VARIATION

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ABSTRACT. In this paper we present results on convergence and Cesàro summability of Multiple Fourier series of functions of bounded generalized variation.

1. CLASSES OF FUNCTIONS OF TWO VARIABLES OF BOUNDED GENERALIZED VARIATION

In 1881 Jordan [20] introduced a class of functions of bounded variation and applied it to the theory of Fourier series. This notion was generalized hereinafter by many authors (quadratic variation, Φ -variation, Λ -variation etc., see [27]-[4]). In two dimensional case the class BV of functions of bounded variation was introduced by Hardy [19].

In this section we introduce several classes of bivariate functions of Bounded Generalized Variation and compare them with the class HBV (see Definition 1.1 below), which is important for the applications in Fourier analysis (see Theorem S in Section 2.).

Let $f(x, y)$, $(x, y) \in \mathbb{R}^2$ be a real function of two variables of period 2π with respect to each variable. Given intervals $I = (a, b)$, $J = (c, d)$ and points x, y from $T := [0, 2\pi]$ we denote

$$f(I, y) := f(b, y) - f(a, y), \quad f(x, J) = f(x, d) - f(x, c)$$

and

$$f(I, J) := f(a, c) - f(a, d) - f(b, c) + f(b, d).$$

Let $E = \{I_i\}$ be a collection of nonoverlapping intervals from T ordered in arbitrary way and let Ω be the set of all such collections E . Denote by Ω_n set of all collections of n nonoverlapping intervals $I_k \subset T$.

For the sequence of positive numbers $\Lambda = \{\lambda_n\}_{n=1}^{\infty}$ we define

$$\Lambda V_1(f) = \sup_y \sup_{E \in \Omega} \sum_n \frac{|f(I_i, y)|}{\lambda_i} \quad (E = \{I_i\}),$$

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$$\Lambda V_2(f) = \sup_x \sup_{F \in \Omega} \sum_m \frac{|f(x, J_j)|}{\lambda_j} \quad (F = \{J_j\}),$$

$$\Lambda V_{1,2}(f) = \sup_{F, E \in \Omega} \sum_i \sum_j \frac{|f(I_i, J_j)|}{\lambda_i \lambda_j}.$$

Definition 1.1. We say that the function f has Bounded Λ -variation on $T^2 = [0, 2\pi]^2$ and write $f \in \Lambda BV$, if

$$\Lambda V(f) := \Lambda V_1(f) + \Lambda V_2(f) + \Lambda V_{1,2}(f) < \infty.$$

We say that f has Bounded Partial Λ -variation and write $f \in P\Lambda BV$ if

$$P\Lambda V(f) := \Lambda V_1(f) + \Lambda V_2(f) < \infty.$$

If $\lambda_n \equiv 1$ (or if $0 < c < \lambda_n < C < \infty$, $n = 1, 2, \dots$) the classes ΛBV and $P\Lambda BV$ coincide with the Hardy class BV and PBV respectively. Hence it is reasonable to assume that $\lambda_n \rightarrow \infty$ and since the intervals in $E = \{I_i\}$ are ordered arbitrarily, we will suppose, without loss of generality, that the sequence $\{\lambda_n\}$ is increasing. Thus,

$$(1.1) \quad 1 < \lambda_1 \leq \lambda_2 \leq \dots, \quad \lim_{n \rightarrow \infty} \lambda_n = \infty.$$

In the case when $\lambda_n = n$, $n = 1, 2, \dots$ we say *Harmonic Variation* instead of Λ -variation and write H instead of Λ (HBV , $PHBV$, $HV(f)$, etc).

The notion of Λ -variation was introduced by D. Waterman [26] in one dimensional case and A. Sahakian [24] in two dimensional case. The class PBV as well as the class PBV_p (see Definition 1.2) was introduced by U. Goginava in [10].

Definition 1.2. Let Φ -be a strictly increasing continuous function on $[0, +\infty)$ with $\Phi(0) = 0$. We say that the function f has Bounded Partial Φ -variation on T^2 and write $f \in PBV_\Phi$, if

$$V_\Phi^{(1)}(f) := \sup_y \sup_{\{I_i\} \in \Omega_n} \sum_{i=1}^n \Phi(|f(I_i, y)|) < \infty, \quad n = 1, 2, \dots,$$

$$V_\Phi^{(2)}(f) := \sup_x \sup_{\{J_j\} \in \Omega_m} \sum_{j=1}^m \Phi(|f(x, J_j)|) < \infty, \quad m = 1, 2, \dots$$

In the case when $\Phi(u) = u^p$, $p \geq 1$, we say that f has Bounded Partial p -variation and write $f \in PBV_p$.

In the following theorem the necessary and sufficient conditions are obtained for the inclusion $P\Lambda BV \subset HBV$.

Theorem 1.1 (U. Goginava, A. Sahakian [11]). Let $\Lambda = \{\lambda_n\}$ with $\lambda_n = n\gamma_n$ and $\gamma_n \geq \gamma_{n+1} > 0$, $n = 1, 2, \dots$.

1) If

$$(1.2) \quad \sum_{n=1}^{\infty} \frac{\gamma_n}{n} < \infty,$$

then $P\Lambda BV \subset HBV$.

2) If $\gamma_n = O(\gamma_{n^{[1+\delta]}})$ for some $\delta > 0$ and

$$\sum_{n=1}^{\infty} \frac{\gamma_n}{n} = \infty,$$

then $P\Lambda BV \not\subset HBV$.

Corollary 1.1. $PBV \subset HBV$ and $PHBV \not\subset HBV$.

Corollary 1.2. Let Φ and Ψ are conjugate functions in the sense of Yung ($ab \leq \Phi(a) + \Psi(b)$) and let for some $\{\lambda_n\}$ satisfying (1.1),

$$(1.3) \quad \sum_{n=1}^{\infty} \Psi\left(\frac{1}{\lambda_n}\right) < \infty.$$

Then $PBV_{\Phi} \subset HBV$. In particular, $PBV_p \subset HBV$ for any $p > 1$.

Definition 1.3 (U. Goginava [10]). The Partial Modulus of Variation of a function f are the functions $v_1(n, f)$ and $v_2(m, f)$ defined by

$$v_1(n, f) := \sup_y \sup_{\{I_i\} \in \Omega_n} \sum_{i=1}^n |f(I_i, y)|, \quad n = 1, 2, \dots,$$

$$v_2(m, f) := \sup_x \sup_{\{J_k\} \in \Omega_m} \sum_{k=1}^m |f(x, J_k)|, \quad m = 1, 2, \dots$$

For functions of one variable the concept of the modulus of variation was introduced by Chanturia [4].

Theorem 1.2 (U. Goginava, A. Sahakian [11]). Let f be such that

$$\sum_{n=1}^{\infty} \frac{\sqrt{v_j(n, f)}}{n^{3/2}} < \infty, \quad j = 1, 2.$$

Then $f \in HBV$.

Another class of functions of generalized bounded variation was introduced by M. Dyachenko and D. Waterman in [7]. Denoting by Γ the the set of finite collections of nonoverlapping rectangles $A_k := [\alpha_k, \beta_k] \times [\gamma_k, \delta_k] \subset T^2$ they define

$$\Lambda^*V(f) := \sup_{\{A_k\} \in \Gamma} \sum_k \frac{|f(A_k)|}{\lambda_k}.$$

Definition 1.4 (M. Dyachenko, D. Waterman [7]). We say that $f \in \Lambda^*BV$ if

$$\Lambda V(f) := \Lambda V_1(f) + \Lambda V_2(f) + \Lambda^*V(f) < \infty.$$

In [14] we introduced a new classes of functions of generalized bounded variation and investigate the convergence of Fourier series of function of that classes. For the sequence $\Lambda = \{\lambda_n\}_{n=1}^{\infty}$ we define

$$\begin{aligned}\Lambda^\# V_1(f) &= \sup_{\{y_i\} \subset T} \sup_{\{I_i\} \in \Omega} \sum_i \frac{|f(I_i, y_i)|}{\lambda_i}, \\ \Lambda^\# V_2(f) &= \sup_{\{x_j\} \subset T} \sup_{\{J_j\} \in \Omega} \sum_j \frac{|f(x_j, J_j)|}{\lambda_j}.\end{aligned}$$

Definition 1.5 (U. Goginava, A. Sahakian [11]). *We say that $f \in \Lambda^\# BV$, if*

$$\Lambda^\# V(f) := \Lambda^\# V_1(f) + \Lambda^\# V_2(f) < \infty.$$

It is not hard to see, that

$$(1.4) \quad \Lambda^* BV \subset \Lambda^\# BV \subset P\Lambda BV.$$

Obviously, the function $f(x, y) = \text{sign}(x - y)$ belongs to $P\Lambda BV \setminus \Lambda^\# BV$ for any Λ . On the other hand, we have proved the following result.

Theorem 1.3 (U. Goginava, A. Sahakian [14]). *If $\Lambda = \{\lambda_n\}$ and*

$$\limsup_{n \rightarrow \infty} \left(\sum_{k=1}^{n^2} \frac{1}{\lambda_k} \right) \left(\sum_{k=1}^n \frac{1}{\lambda_k} \right)^{-1} = +\infty,$$

then $\Lambda^\# BV \setminus \Lambda^ BV \neq \emptyset$.*

In the next theorem we characterize sequences $\Lambda = \{\lambda_n\}$ for which the inclusion $\Lambda^\# BV \subset HBV$ holds.

Theorem 1.4 (U. Goginava, A. Sahakian [14]). *Let $\Lambda = \{\lambda_n\}$.*

a) If

$$\limsup_{n \rightarrow \infty} \frac{\lambda_n \log n}{n} < \infty,$$

then

$$\Lambda^\# BV \subset HBV.$$

b) If $\frac{\lambda_n}{n} \downarrow 0$ and

$$\limsup_{n \rightarrow \infty} \frac{\lambda_n \log n}{n} = +\infty,$$

then

$$\Lambda^\# BV \not\subset HBV.$$

Definition 1.6 (U. Goginava, A. Sahakian [14]). *Let Φ -be a strictly increasing continuous function on $[0, +\infty)$ with $\Phi(0) = 0$. We say that the function $f \in B^\# V_\Phi(T^2)$, if*

$$V_{\Phi,1}^\#(f) := \sup_{\{y_i\} \subset T} \sup_{\{I_i\} \in \Omega} \sum_i \Phi(|f(I_i, y_i)|) < \infty,$$

and

$$V_{\Phi,2}^{\#}(f) := \sup_{\{x_j\} \subset T} \sup_{\{J_j\} \in \Omega} \sum_j \Phi(|f(x_j, J_j)|) < \infty.$$

Next, we define

$$v_1^{\#}(n, f) := \sup_{\{y_i\}_{i=1}^n} \sup_{\{I_i\} \in \Omega_n} \sum_{i=1}^n |f(I_i, y_i)|, \quad n = 1, 2, \dots,$$

$$v_2^{\#}(m, f) := \sup_{\{x_j\}_{j=1}^m} \sup_{\{J_k\} \in \Omega_m} \sum_{j=1}^m |f(x_j, J_j)|, \quad m = 1, 2, \dots$$

Theorem 1.5 (U. Goginava, A. Sahakian [14]). *Let Φ and Ψ are conjugate functions in the sense of Yung ($ab \leq \Phi(a) + \Psi(b)$) and let*

$$\sum_{n=1}^{\infty} \Psi\left(\frac{\log n}{n}\right) < \infty.$$

Then

$$B^{\#}V_{\Phi} \subset \left\{\frac{n}{\log n}\right\}^{\#} BV.$$

Theorem 1.6 (U. Goginava, A. Sahakian [14]). *Let*

$$\sum_{n=1}^{\infty} \frac{v_s^{\#}(f, n) \log n}{n^2} < \infty, \quad s = 1, 2.$$

Then

$$f \in \left\{\frac{n}{\log n}\right\}^{\#} BV.$$

Observe that by Theorem 1.4 we have the inclusion $\left\{\frac{n}{\log n}\right\}^{\#} BV \subset HBV$. Now, for a sequence $\Lambda = \{\lambda_n\}$ we denote

$$\Lambda_n := \{\lambda_k\}_{k=n}^{\infty}, \quad n = 1, 2, \dots$$

Definition 1.7 (U Goginava [12]). *We say that the function $f \in \Lambda^{\#}BV$ is continuous in $\Lambda^{\#}$ -variation and write $f \in C\Lambda^{\#}V$, if*

$$\lim_{n \rightarrow \infty} \Lambda_n^{\#}V_1(f) = \lim_{n \rightarrow \infty} \Lambda_n^{\#}V_2(f) = 0.$$

Theorem 1.7 (U. Goginava, A. Sahakian [17]). *Let the sequence $\Lambda = \{\lambda_n\}$ be such that*

$$\liminf_{n \rightarrow \infty} \frac{\lambda_{2n}}{\lambda_n} = q > 1.$$

Then $\Lambda^{\#}BV = C\Lambda^{\#}V$.

Theorem 1.8 (U. Goginava [12]). *Let $\alpha + \beta < 1, \alpha, \beta > 0$ and*

$$\sum_{j=1}^{\infty} \frac{v_s^{\#}(f; 2^j)}{2^{j(1-(\alpha+\beta))}} < \infty, \quad s = 1, 2.$$

Then $f \in C\{n^{1-(\alpha+\beta)}\}^{\#}V$.

2. CONVERGENCE OF DOUBLE FOURIER SERIES

Everywhere in this and in the next section we suppose that the function f is measurable on \mathbb{R}^2 and 2π -periodic with respect to each variable. The double Fourier series of a function $f \in L^1(T^2)$ with respect to the trigonometric system is the series

$$S[f] := \sum_{m,n=-\infty}^{+\infty} \widehat{f}(m,n) e^{imx} e^{iny},$$

where

$$\widehat{f}(m,n) = \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} f(x,y) e^{-imx} e^{-iny} dx dy$$

are the Fourier coefficients of f . The rectangular partial sums of $S[f]$ are defined as follows:

$$S_{M,N}[f, (x,y)] := \sum_{m=-M}^M \sum_{n=-N}^N \widehat{f}(m,n) e^{imx} e^{iny},$$

In this paper we consider only **Pringsheim convergence** of double Fourier series, i.e **convergence of rectangular partial sums** $S_{M,N}[f, (x,y)]$, as $M, N \rightarrow \infty$.

We denote by $C(T^2)$ the space of continuous on \mathbb{R}^2 and 2π -periodic with respect to each variable functions with the norm

$$\|f\|_C := \sup_{x,y \in T^2} |f(x,y)|.$$

For a function f we denote by $f(x \pm 0, y \pm 0)$ the open coordinate quadrant limits (if exist) at the point (x,y) and let $f^*(x,y)$ be the arithmetic mean of that quadrant limits:

$$(2.1) \quad f^*(x,y) := \frac{1}{4} \{f(x+0, y+0) + f(x+0, y-0) + f(x-0, y+0) + f(x-0, y-0)\}.$$

Remark 2.1. *Observe that for a function $f \in \Lambda BV$ the quadrant limits $f(x \pm 0, y \pm 0)$ may not exist. As was shown in [14] for any function $f \in \Lambda^{\#}BV$ the quadrant limits $f(x \pm 0, y \pm 0)$ exist at any point $(x,y) \in T^2$.*

We say the point $(x,y) \in T^2$ is a **regular point** of a function f , if all quadrant limits in (2.1) exist.

The well known Dirichlet-Jordan theorem (see [29]) states that the Fourier series of a function $g(x)$, $x \in T$ of bounded variation converges at every point

x to the value $[g(x+0) + g(x-0)]/2$. If g is in addition continuous on T the Fourier series converges uniformly on T .

Hardy [19] generalized the Dirichlet-Jordan theorem to the double Fourier series. He proved that if function f has bounded variation in the sense of Hardy ($f \in BV$), then $S[f]$ converges to $f^*(x, y)$ at any regular point (x, y) . If f is in addition continuous on T^2 then $S[f]$ converges uniformly on T^2 .

Theorem S (Sahakian [24]). *The Fourier series of a function $f \in HBV$ converges to $f^*(x, y)$ in any regular point (x, y) . The convergence is uniform on any compact $K \subset T^2$, where the function f is continuous.*

Theorem S was proved in [24] under assumption that the function is continuous on some open set containing K , while O. Sargsyan noticed in [23], that the continuity of f on the compact K is sufficient.

Definition 2.1. *We say that the class of functions $V \subset L^1(T^2)$ is a class of convergence on T^2 , if for any function $f \in V$*

- 1) *the Fourier series of f converges to $f^*(x, y)$ at any regular point (x, y) ,*
- 2) *the convergence is uniform on any compact $K \subset T^2$, where the function f is continuous.*

The following results immediately follow from Theorems 1.1, 1.2, Corollary 1.2 and Theorem S.

Theorem 2.1 (U. Goginava, A. Sahakian [11]). *Let $\Lambda = \{\lambda_n\}$ with $\lambda_n = n\gamma_n$ and $\gamma_n \geq \gamma_{n+1} > 0$, $n = 1, 2, \dots$.*

- 1) *If*

$$\sum_{n=1}^{\infty} \frac{\gamma_n}{n} < \infty,$$

then the class $P\Lambda BV$ is a class of convergence on T^2 .

- 2) *If $\gamma_n = O(\gamma_{n[1+\delta]})$ for some $\delta > 0$ and*

$$\sum_{n=1}^{\infty} \frac{\gamma_n}{n} = \infty,$$

then there exists a continuous function $f \in P\Lambda BV$, the Fourier series of which diverges over cubes at $(0, 0)$.

Theorem 2.2 (U. Goginava, A. Sahakian [11]). *The set of functions f satisfying*

$$\sum_{n=1}^{\infty} \frac{\sqrt{v_j(n, f)}}{n^{3/2}} < \infty, \quad j = 1, 2,$$

is a class of convergence on T^2 .

Corollary 2.1. *The set of functions f satisfying $v_1(n, f) = O(n^\alpha)$, $v_2(n, f) = O(n^\beta)$, $0 < \alpha, \beta < 1$, is a class of convergence on T^2 .*

Theorem 2.3 (U. Goginava [10]). *The class PBV_p , $p \geq 1$, is a class of convergence on T^2 .*

From Theorem 2.1 follows that for any $\delta > 0$ the class $f \in P \left\{ \frac{n}{\log^{1+\delta} n} \right\} BV$ is a class of convergence. Moreover, one can not take here $\delta = 0$. It is interesting to compare this result with the following one obtained by M. Dyachenko and D. Waterman in [7].

Theorem DW (M. Dyachenko and D. Waterman [7]). *If $f \in \left\{ \frac{n}{\log n} \right\}^* BV$, then in any point $(x, y) \in T^2$ the quadrant limits (2.1) exist and the double Fourier series of f converges to $f^*(x, y)$. Moreover, the sequence $\left\{ \frac{n}{\log n} \right\}$ can not be replaced with any sequence $\left\{ \frac{n\alpha_n}{\log n} \right\}$, where $\alpha_n \rightarrow \infty$.*

It is easy to show (see [7]), that $\left\{ \frac{n}{\log n} \right\}^* BV \subset HBV$, hence the convergence part of Theorem DW follows from Theorem S. It is essential that the condition $f \in \left\{ \frac{n}{\log n} \right\}^* BV$ guaranties the existence of quadrant limits.

The following theorem immediately follows from Theorem 1.4 and Theorem S.

Theorem 2.4 (U. Goginava, A. Sahakian [14]). *If $\Lambda = \{\lambda_n\}$ and*

$$\limsup_{n \rightarrow \infty} \frac{\lambda_n \log n}{n} < \infty,$$

then the class $\Lambda^\# BV$ is a class of convergence on T^2 .

In particular, the class $\left\{ \frac{n}{\log n} \right\}^\# BV$ is a class of convergence on T^2 .

Theorem DW and (1.4) imply that the sequence $\left\{ \frac{n}{\log n} \right\}$ in Theorem 2.4 can not be replaced with any sequence $\left\{ \frac{n\alpha_n}{\log n} \right\}$, where $\alpha_n \rightarrow \infty$.

Theorems 1.5, 1.6 and 2.4 imply

Theorem 2.5 (U. Goginava, A. Sahakian [14]). *The class $B^\# V_\Phi$ is a class of convergence on T^2 , provided that (1.2) and (1.3) hold.*

Theorem 2.6 (U. Goginava, A. Sahakian [14]). *Let*

$$\sum_{n=1}^{\infty} \frac{v_s^\#(f, n) \log n}{n^2} < \infty, \quad s = 1, 2.$$

Then in any point $(x, y) \in T^2$ the quadrant limits (2.1) exist and the double Fourier series of f converges to $f^(x, y)$. The convergence is uniform on any compact $K \in T^2$, if f is continuous on K .*

3. CESÀRO SUMMABILITY OF DOUBLE FOURIER SERIES

For one-dimensional Fourier series D. Waterman has proved the following theorem.

Theorem W2 (D. Waterman [25]). *Let $0 < \alpha < 1$. The Fourier series of a function $f \in \{n^{1-\alpha}\}BV$ is everywhere $(C, -\alpha)$ bounded and is uniformly $(C, -\alpha)$ bounded on each closed interval of continuity of f .*

If $f \in C\{n^{1-\alpha}\}BV$, then $S[f]$ is everywhere $(C, -\alpha)$ summable to the value $[f(x+0) + f(x-0)]/2$ and the summability is uniform on each closed interval of continuity.

Later A. Sablin proved in [22], that for $0 < \alpha < 1$ the classes $\{n^{1-\alpha}\}BV$ and $C\{n^{1-\alpha}\}BV$ coincide.

For double Fourier series the **Cesàro $(C; \alpha, \beta)$ -means** of a function $f \in L^1(T^2)$ are defined by

$$\sigma_{n,m}^{\alpha,\beta}(f; x, y) := \frac{1}{A_n^\alpha} \frac{1}{A_m^\beta} \sum_{i=0}^n \sum_{j=0}^m A_{n-i}^{\alpha-1} A_{m-j}^{\beta-1} S_{i,j}[f, (x, y)],$$

where $\alpha, \beta > -1$ and

$$A_0^\alpha = 1, \quad A_k^\alpha = \frac{(\alpha+1) \cdots (\alpha+k)}{k!}, \quad k = 1, 2, \dots$$

The double Fourier series of f is said to be $(C; \alpha, \beta)$ **summable to s in a point (x, y)** , if

$$\lim_{n,m \rightarrow \infty} \sigma_{n,m}^{\alpha,\beta}(f; x, y) = s.$$

L. Zhizhiashvili has investigated the convergence of Cesàro means of double Fourier series of functions of bounded variation. In particular, the following theorem was proved.

Theorem Zh (L. Zhizhiashvili [28]). *If $f \in BV$, then the double Fourier series of f is $(C; -\alpha, -\beta)$ summable to $f^*(x, y)$ in any regular point (x, y) . The convergence is uniform on any compact K , where the function f is continuous.*

For functions of partial bounded variation the problem was considered by the first author.

Theorem G2 (U. Goginava [8]). *Let $\alpha > 0, \beta > 0$.*

- 1) *If $\alpha + \beta < 1$, then for any $f \in C(T^2) \cap PBV$ the double Fourier series of f is uniformly $(C; -\alpha, -\beta)$ summable to f .*
- 2) *If $\alpha + \beta \geq 1$, then there exists a continuous function $f_0 \in PBV$ such that the sequence $\sigma_{n,n}^{-\alpha,-\beta}(f_0; 0, 0)$ diverges.*

In [13] we consider the following problem. *Let $\alpha, \beta \in (0, 1), \alpha + \beta < 1$. Under what conditions on the sequence $\Lambda = \{\lambda_n\}$ the double Fourier series of any function $f \in P\Lambda BV$ is $(C; -\alpha, -\beta)$ summable?*

Theorem 3.1 (U. Goginava, A. Sahakian [13]). *Let $\alpha, \beta \in (0, 1), \alpha + \beta < 1$ and let the sequence $\Lambda = \{\lambda_k\}$ be such that $\lambda_k k^{(\alpha+\beta)-1} \downarrow 0$.*

- 1) *If*

$$\sum_{k=1}^{\infty} \frac{\lambda_k}{k^{2-(\alpha+\beta)}} < \infty,$$

then the double Fourier series of any function $f \in P\Lambda BV$ is $(C; -\alpha, -\beta)$ summable to $f^*(x, y)$ at any regular point (x, y) . The summability is uniform on any compact K , if f is continuous on the neighborhood of K .

2) If

$$\sum_{k=1}^{\infty} \frac{\lambda_k}{k^{2-(\alpha+\beta)}} = \infty,$$

then there exists a continuous function $f \in P\Lambda BV$ for which the $(C; -\alpha, -\beta)$ means of the double Fourier series diverges over cubes at $(0, 0)$.

Corollary 3.1 (U. Goginava, A. Sahakian [13]). *Let $\alpha, \beta \in (0, 1)$, $\alpha + \beta < 1$.*

1) *If $f \in P\left\{\frac{n^{1-(\alpha+\beta)}}{\log^{1+\varepsilon} n}\right\} BV$ for some $\varepsilon > 0$, then the double Fourier series of the function f is $(C; -\alpha, -\beta)$ summable to $f^*(x, y)$ in any regular point (x, y) . The summability is uniform on any compact K , if f is continuous on the neighborhood of K .*

2) *There exists a continuous function $f \in P\left\{\frac{n^{1-(\alpha+\beta)}}{\log(n+1)}\right\} BV$ such that $(C; -\alpha, -\beta)$ means of two-dimensional Fourier series of f diverges over cubes at $(0, 0)$.*

Corollary 3.2 (U. Goginava, A. Sahakian [13]). *Let $\alpha, \beta \in (0, 1)$, $\alpha + \beta < 1$ and $f \in PBV$. Then the double Fourier series of the function f is $(C; -\alpha, -\beta)$ summable to $f^*(x, y)$ in any regular point (x, y) . The summability is uniform on any compact K , if f is continuous on the neighborhood of K .*

In [12] the following problem was considered. Let $\alpha, \beta \in (0, 1)$, $\alpha + \beta < 1$. Under what conditions on the sequence $\Lambda = \{\lambda_n\}$ the double Fourier series of any function $f \in C\Lambda^\# BV$ is $(C; -\alpha, -\beta)$ summable.

Theorem 3.2 (U. Goginava [12]). *a) Let $\alpha, \beta \in (0, 1)$, $\alpha + \beta < 1$ and $f \in C\{n^{1-(\alpha+\beta)}\}^\# BV$. Then the double Fourier series of f is $(C; -\alpha, -\beta)$ summable to $f^*(x, y)$ in any point (x, y) . The summability is uniform on any compact $K \subset \mathbb{T}^2$, if f is continuous on the neighborhood of K .*

b) Let $\Lambda := \{n^{1-(\alpha+\beta)}\xi_n\}$, where $\xi_n \uparrow \infty$ as $n \rightarrow \infty$. Then there exists a function $f \in C(\mathbb{T}^2) \cap C\Lambda^\# V$ for which $(C; -\alpha, -\beta)$ -means of double Fourier series diverges unboundedly at $(0, 0)$.

Theorems 1.7, 1.8 and 3.2 imply the following results.

Theorem 3.3. *Let $\alpha, \beta \in (0, 1)$, $\alpha + \beta < 1$ and $f \in \{n^{1-(\alpha+\beta)}\}^\# BV$. Then the double Fourier series of f is $(C; -\alpha, -\beta)$ summable to $f^*(x, y)$ in any point (x, y) . The summability is uniform on any compact $K \subset \mathbb{T}^2$, if f is continuous on the neighborhood of K .*

Theorem 3.4. *Let $\alpha, \beta \in (0, 1)$, $\alpha + \beta < 1$ and*

$$\sum_{j=1}^{\infty} \frac{v_s^\#(f; 2^j)}{2^{j(1-(\alpha+\beta))}} < \infty, \quad s = 1, 2.$$

Then the double Fourier series of f is $(C; -\alpha, -\beta)$ summable to $f^*(x, y)$ in any point (x, y) . The summability is uniform on any compact $K \subset \mathbb{T}^2$, if f is continuous on the neighborhood of K .

4. CLASSES OF FUNCTIONS OF d VARIABLES OF BOUNDED GENERALIZED VARIATION

Consider a function $f(x)$ defined on the d -dimensional cube T^d and a collection of intervals

$$J^k = (a^k, b^k) \subset T, \quad k = 1, 2, \dots, d.$$

For $d = 1$ we set

$$f(J^1) := f(b^1) - f(a^1).$$

If for any function of $d - 1$ variables the expression $f(J^1 \times \dots \times J^{d-1})$ is already defined, then for a function f of d variables the **mixed difference** is defined as follows:

$$f(J^1 \times \dots \times J^d) := f(J^1 \times \dots \times J^{d-1}, b^d) - f(J^1 \times \dots \times J^{d-1}, a^d).$$

For sequences of positive numbers

$$\Lambda^j = \{\lambda_n^j\}_{n=1}^\infty, \quad \lim_{n \rightarrow \infty} \lambda_n^j = \infty, \quad j = 1, 2, \dots, d,$$

and for a function $f(x)$, $x = (x_1, \dots, x_d) \in T^d$ the $(\Lambda^1, \dots, \Lambda^d)$ -**variation of f with respect to the index set $D := \{1, 2, \dots, d\}$** is defined as follows:

$$\{\Lambda^1, \dots, \Lambda^d\} V^D(f, T^d) := \sup_{\{I_{i_j}^j\} \in \Omega} \sum_{i_1, \dots, i_d} \frac{|f(I_{i_1}^1 \times \dots \times I_{i_d}^d)|}{\lambda_{i_1}^1 \dots \lambda_{i_d}^d}.$$

For an index set $\alpha = \{j_1, \dots, j_p\} \subset D$ and any $x = (x_1, \dots, x_d) \in R^d$ we set $\tilde{\alpha} := D \setminus \alpha$ and denote by x_α the vector of R^p consisting of components $x_j, j \in \alpha$, i.e.

$$x_\alpha = (x_{j_1}, \dots, x_{j_p}) \in R^p.$$

By

$$\{\Lambda^{j_1}, \dots, \Lambda^{j_p}\} V^\alpha(f, x_{\tilde{\alpha}}, T^d) \quad \text{and} \quad f(I_{i_{j_1}}^1 \times \dots \times I_{i_{j_p}}^p, x_{\tilde{\alpha}})$$

we denote respectively the $(\Lambda^{j_1}, \dots, \Lambda^{j_p})$ -variation over the p -dimensional cube T^p and mixed difference of f as a function of variables x_{j_1}, \dots, x_{j_p} with fixed values $x_{\tilde{\alpha}}$ of other variables. The $(\Lambda^{j_1}, \dots, \Lambda^{j_p})$ -**variation of f with respect to the index set α** is defined as follows:

$$\{\Lambda^{j_1}, \dots, \Lambda^{j_p}\} V^\alpha(f, T^p) = \sup_{x_{\tilde{\alpha}} \in T^{d-p}} \{\Lambda^{j_1}, \dots, \Lambda^{j_p}\} V^\alpha(f, x_{\tilde{\alpha}}, T^d).$$

Definition 4.1. We say that the function f has total Bounded $(\Lambda^1, \dots, \Lambda^d)$ -variation on T^d and write $f \in \{\Lambda^1, \dots, \Lambda^d\} BV(T^d)$, if

$$\{\Lambda^1, \dots, \Lambda^d\} V(f, T^d) := \sum_{\alpha \subset D} \{\Lambda^1, \dots, \Lambda^d\} V^\alpha(f, T^d) < \infty.$$

Definition 4.2. We say that the function f is continuous in $(\Lambda^1, \dots, \Lambda^d)$ -variation on T^d and write $f \in C\{\Lambda^1, \dots, \Lambda^d\} V(T^d)$, if

$$\lim_{n \rightarrow \infty} \{\Lambda^{j_1}, \dots, \Lambda^{j_{k-1}}, \Lambda_n^{j_k}, \Lambda^{j_{k+1}}, \dots, \Lambda^{j_p}\} V^\alpha(f, T^d) = 0, \quad k = 1, 2, \dots, p$$

for any $\alpha \subset D$, $\alpha := \{j_1, \dots, j_p\}$, where $\Lambda_n^{j_k} := \left\{ \lambda_s^{j_k} \right\}_{s=n}^\infty$.

The continuity of a function in Λ -variation was introduced by D. Waterman [25] and was investigated in details by A. Bakhvalov (see [1], [2] and references therein). This property is important for applications in the theory of Fourier series (see Theorem B1 in Section 5).

Definition 4.3. We say that the function f has Bounded Partial $(\Lambda^1, \dots, \Lambda^d)$ -variation and write $f \in P\{\Lambda^1, \dots, \Lambda^d\} BV(T^d)$ if

$$P\{\Lambda^1, \dots, \Lambda^d\} V(f, T^d) := \sum_{i=1}^d \Lambda^i V^{\{i\}}(f, T^d) < \infty.$$

In the case when $\Lambda^1 = \dots = \Lambda^d = \Lambda$ we set

$$\begin{aligned} \Lambda BV(T^d) &:= \{\Lambda^1, \dots, \Lambda^d\} BV(T^d), \\ C\Lambda V(T^d) &:= C\{\Lambda^1, \dots, \Lambda^d\} V(T^d), \\ P\Lambda BV(T^d) &:= P\{\Lambda^1, \dots, \Lambda^d\} BV(T^d). \end{aligned}$$

If $\lambda_n = n$ for all $n = 1, 2, \dots$ we say *Harmonic Variation* instead of Λ -variation and write H instead of Λ , i.e. HBV , $PHBV$, CHV , etc.

Theorem 4.1 (U. Goginava, A. Sahakian [15]). Let $\Lambda = \{\lambda_n\}_{n=1}^\infty$ and $d \geq 2$. If $\lambda_n/n \downarrow 0$ and

$$\sum_{n=1}^\infty \frac{\lambda_n \log^{d-2} n}{n^2} < \infty,$$

then $P\Lambda BV(T^d) \subset CHV(T^d)$.

For a sequence $\Lambda = \{\lambda_n\}_{n=1}^\infty$ we denote

$$\Lambda^\# V_s(f, T^d) := \sup_{\{x^i\{s\}\} \subset T^{d-1}} \sup_{\{I_i^s\} \in \Omega} \sum_i \frac{|f(I_i^s, x^i\{s\})|}{\lambda_i},$$

where

$$x^i\{s\} := (x_1^i, \dots, x_{s-1}^i, x_{s+1}^i, \dots, x_d^i) \quad \text{for} \quad x^i := (x_1^i, \dots, x_d^i).$$

Definition 4.4. We say that $f \in \Lambda^\# BV(T^d)$, if

$$\Lambda^\# V(f, T^d) := \sum_{s=1}^d \Lambda^\# V_s(f, T^d) < \infty.$$

Theorem 4.2 (U. Goginava, A. Sahakian [18]). If $\Lambda = \{\lambda_n\}$ with

$$\lambda_n = \frac{n}{\log^{d-1} n}, \quad n = 2, 3, \dots,$$

then $\Lambda^\# BV(T^d) \subset HBV(T^d)$.

Now, we denote

$$\Delta := \{\delta = (\delta_1, \dots, \delta_d) : \delta_i = \pm 1, i = 1, 2, \dots, d\}$$

and

$$\pi_{\varepsilon\delta}(x) := (x_1, x_1 + \varepsilon\delta_1) \times \dots \times (x_d, x_d + \varepsilon\delta_d),$$

for $x = (x_1, \dots, x_d) \in R^d$ and $\varepsilon > 0$. We set $\pi_\delta(x) := \pi_{\varepsilon\delta}(x)$, if $\varepsilon = 1$.

For a function f and $\delta \in \Delta$ we set

$$(4.1) \quad f_\delta(x) := \lim_{t \in \pi_\delta(x), t \rightarrow x} f(t),$$

if the last limit exists.

Theorem 4.3 (U. Goginava, A. Sahakian [18]). Suppose $\Lambda = \{\lambda_n\}$ and $f \in \Lambda^\# BV(T^d)$.

a) If the limit $f_\delta(x)$ exists for some $x = (x_1, \dots, x_d) \in T^d$ and some $\delta = (\delta_1, \dots, \delta_d) \in \Delta$, then

$$\lim_{\varepsilon \rightarrow 0} \Lambda^\# V(f, \pi_{\varepsilon\delta}(x)) = 0.$$

b) If f is continuous on some compact $K \subset T^d$, then

$$\lim_{\varepsilon \rightarrow 0} \Lambda^\# V(f, [x_1 - \varepsilon, x_1 + \varepsilon] \times \dots \times [x_d - \varepsilon, x_d + \varepsilon]) = 0$$

uniformly with respect to $x = (x_1, \dots, x_d) \in K$.

Theorem 4.4 (U. Goginava, A. Sahakian [18]). If the function $f(x)$, $x \in T^d$ satisfies the condition

$$\sum_{n=1}^{\infty} \frac{v_s^\#(f, n) \log^{d-1} n}{n^2} < \infty, \quad s = 1, 2, \dots, d,$$

then $f \in \left\{ \frac{n}{\log^{d-1} n} \right\}^\# BV(T^d)$.

5. CONVERGENCE OF MULTIPLE FOURIER SERIES

The Fourier series of function $f \in L^1(T^d)$ with respect to the trigonometric system is the series

$$S[f] := \sum_{n_1, \dots, n_d = -\infty}^{+\infty} \widehat{f}(n_1, \dots, n_d) e^{i(n_1 x_1 + \dots + n_d x_d)},$$

where

$$\widehat{f}(n_1, \dots, n_d) = \frac{1}{(2\pi)^d} \int_{T^d} f(x_1, \dots, x_d) e^{-i(n_1 x_1 + \dots + n_d x_d)} dx_1 \dots dx_d$$

are the Fourier coefficients of f . The rectangular partial sums are defined as follows:

$$S_{N_1, \dots, N_d}[f, (x_1, \dots, x_d)] = \sum_{n_1 = -N_1}^{N_1} \dots \sum_{n_d = -N_d}^{N_d} \widehat{f}(n_1, \dots, n_d) e^{i(n_1 x_1 + \dots + n_d x_d)}$$

We denote by $C(T^d)$ the space of continuous and 2π -periodic with respect to each variable functions with the norm

$$\|f\|_C := \sup_{(x^1, \dots, x^d) \in T^d} |f(x^1, \dots, x^d)|.$$

We say that the point $x := (x^1, \dots, x^d) \in T^d$ is a **regular point** of a function f if the limits (4.1) exist for all $\delta \in \Delta$. For a regular point $x \in T^d$ we denote

$$f^*(x) := \frac{1}{2^d} \sum_{\delta \in \Delta} f_\delta(x).$$

Definition 5.1. We say that the class of functions $V \subset L^1(T^d)$ is a class of convergence on T^d , if for any function $f \in V$

- 1) the Fourier series of f converges to $f^*(x)$ at any regular point $x \in T^d$,
- 2) the convergence is uniform on any compact $K \subset T^d$, if f is continuous on the neighborhood of K .

In [1] A. Bakhvalov showed that the class $HBV(T^d)$ is not a class of convergence on T^d , if $d > 2$. On the other hand, he proved the following

Theorem B1 (A. Bakhvalov [1]). *The class $CHV(T^d)$ is a class of convergence on T^d for any $d = 1, 2, \dots$*

Convergence of spherical and other partial sums of d -dimensional Fourier series of functions of bounded Λ -variation was investigated in details by M. Dyachenko [5, 6], A. Bakhvalov [1, 3].

The first part of the next theorem is a consequence of Theorem 4.1 and Theorem B1.

Theorem 5.1 ([U. Goginava, A. Sahakian [15]]). *Let $\Lambda = \{\lambda_n\}$ and $d \geq 2$.*

a) If $\lambda_n/n \downarrow 0$ and

$$\sum_{n=1}^{\infty} \frac{\lambda_n \log^{d-2} n}{n^2} < \infty,$$

then $P\Lambda BV$ is a class of convergence on T^d .

b) If $\frac{\lambda_n}{n} = O\left(\frac{\lambda_{[n^\delta]}}{[n^\delta]}\right)$ for some $\delta > 1$, and

$$\sum_{n=1}^{\infty} \frac{\lambda_n \log^{d-2} n}{n^2} = \infty,$$

then there exists a continuous function $f \in P\Lambda BV$, the Fourier series of which diverges at $(0, \dots, 0)$.

Theorem 5.1 imply

Corollary 5.1. *a) If $\Lambda = \{\lambda_n\}_{n=1}^{\infty}$ with*

$$\lambda_n = \frac{n}{\log^{d-1+\varepsilon} n}, \quad n = 2, 3, \dots$$

for some $\varepsilon > 0$, then the class $P\Lambda BV$ is a class of convergence on T^d .

b) If $\Lambda = \{\lambda_n\}_{n=1}^{\infty}$ with

$$\lambda_n = \frac{n}{\log^{d-1} n}, \quad n = 2, 3, \dots,$$

then the class $P\Lambda BV$ is not a class of convergence on T^d .

Theorem 5.2 (Goginava, Sahakian [18]). *a) If $\Lambda = \{\lambda_n\}_{n=1}^{\infty}$ with*

$$\lambda_n = \frac{n}{\log^{d-1} n}, \quad n = 2, 3, \dots,$$

then the class $\Lambda^{\#}BV(T^d)$ is a class of convergence on T^d .

b) If $\Lambda = \{\lambda_n\}_{n=1}^{\infty}$ with

$$\lambda_n := \left\{ \frac{n\xi_n}{\log^{d-1} n} \right\}, \quad n = 2, 3, \dots,$$

where $\xi_n \rightarrow \infty$ as $n \rightarrow \infty$, then there exists a continuous function $f \in \Lambda^{\#}BV(T^d)$ such that the cubical partial sums of d -dimensional Fourier series of f diverge unboundedly at $(0, \dots, 0) \in T^d$.

Theorem 5.3 (Goginava, Sahakian [18]). *For any $d > 1$ the class of functions $f(x)$, $x \in T^d$ satisfying the following condition*

$$\sum_{n=1}^{\infty} \frac{v_s^{\#}(f, n) \log^{d-1} n}{n^2} < \infty, \quad s = 1, \dots, d,$$

is a class of convergence.

6. CESÀRO SUMMABILITY OF d -DIMENSIONAL FOURIER SERIES

The Cesàro $(C; \alpha_1, \dots, \alpha_d)$ means of d -dimensional Fourier series of function $f \in L^1(T^d)$ is defined by

$$\begin{aligned} \sigma_{m_1, \dots, m_d}^{\alpha_1, \dots, \alpha_d}[f; (x_1, \dots, x_d)] \\ := \left(\prod_{i=1}^d A_{m_i}^{\alpha_i} \right)^{-1} \sum_{p_1=0}^{m_1} \cdots \sum_{p_d=0}^{m_d} \prod_{i=1}^d A_{m_i-p_i}^{\alpha_i-1} S_{p_1, \dots, p_d}[f, (x_1, \dots, x_d)], \end{aligned}$$

where

$$A_0^\alpha = 1, \quad A_n^\alpha = \frac{(\alpha+1) \cdots (\alpha+n)}{n!}, \quad \alpha > -1.$$

The Fourier series $S[f]$ is said to be $(C; -\alpha_1, \dots, -\alpha_d)$ summable to s in a point (x_1, \dots, x_d) , if

$$\sigma_{m_1, \dots, m_d}^{\alpha_1, \dots, \alpha_d}[f; (x_1, \dots, x_d)] \rightarrow s \quad \text{as } x_1, \dots, x_d \rightarrow \infty.$$

Definition 6.1. We say that the class of functions $\Omega \subset L^1(T^d)$ is a class of $(C; -\alpha_1, \dots, -\alpha_d)$ summability on T^d , if the Cesaro $(C; -\alpha_1, \dots, -\alpha_d)$ means of Fourier series of any function $f \in \Omega$ converges to $f^*(x)$ at any regular point $x \in T^d$. The summability is uniform on any compact $K \subset T^d$, if in addition, f is continuous on the neighborhood of K .

The multivariate analog of Theorem W2 from Section 3 was proved by A. Bakhvalov in [2].

Theorem B2 (A. Bakhvalov [2]). For any numbers $\alpha_1, \dots, \alpha_d \in (0, 1)$ the class $C\{n^{1-\alpha_1}, \dots, n^{1-\alpha_d}\}V(T^d)$ is a class of $(C; -\alpha_1, \dots, -\alpha_d)$ summability on T^d .

In the next theorem we consider the problem of $(C; -\alpha_1, \dots, -\alpha_d)$ summability of the Fourier series of functions of bounded partial Λ -variation.

Theorem 6.1 (U. Goginava, A. Sahakian [16]). Suppose $\alpha_1, \dots, \alpha_d \in (0, 1)$, $\alpha_1 + \dots + \alpha_d < 1$ and the sequence $\Lambda = \{\lambda_n\}_{n=1}^\infty$ is such that

$$\frac{\lambda_n}{n^{1-(\alpha_1+\dots+\alpha_d)}} \downarrow 0.$$

a) If

$$\sum_{n=1}^\infty \frac{\lambda_n}{n^{2-(\alpha_1+\dots+\alpha_d)}} < \infty,$$

then $P\Lambda BV(T^d)$ is a class of $(C; -\alpha_1, \dots, -\alpha_d)$ summability on T^d .

b) If

$$\sum_{n=1}^\infty \frac{\lambda_n}{n^{2-(\alpha_1+\dots+\alpha_d)}} = \infty,$$

then there exists a continuous function $f \in P\Lambda BV(T^d)$ for which the sequence $\sigma_{N, \dots, N}^{-\alpha_1, \dots, -\alpha_d}[f, (0, \dots, 0)]$ diverges.

Corollary 6.1 (U. Goginava, A. Sahakian [16]). *Suppose $\alpha_1, \dots, \alpha_d \in (0, 1)$, $\alpha_1 + \dots + \alpha_d < 1$ and $\Lambda = \{\lambda_n\}_{n=1}^\infty$.*

a) *If*

$$\lambda_n = \frac{n^{1-(\alpha_1+\dots+\alpha_d)}}{\log^{1+\varepsilon} n}, \quad n = 2, 3, \dots$$

for some $\varepsilon > 0$, then the class $P\Lambda BV(T^d)$ is a class of $(C; -\alpha_1, \dots, -\alpha_d)$ summability on T^d .

b) *If*

$$\lambda_n = \frac{n^{1-(\alpha_1+\dots+\alpha_d)}}{\log n}, \quad n = 2, 3, \dots,$$

then $P\Lambda BV(T^d)$ is not a class of $(C; -\alpha_1, \dots, -\alpha_d)$ summability on T^d .

Theorem 6.2 (U. Goginava, A. Sahakian [16]). *Let $\alpha_1, \dots, \alpha_d \in (0, 1)$, $\alpha_1 + \dots + \alpha_d < 1$. Then the set of functions f satisfying the conditions*

$$\sum_{j=0}^{\infty} \frac{(v_i(2^j, f))^{\alpha_i/(\alpha_1+\dots+\alpha_d)}}{2^{j(\alpha_i/(\alpha_1+\dots+\alpha_d)-\alpha_i)}} < \infty \quad \text{for } i = 1, \dots, d,$$

is a class of $(C; -\alpha_1, \dots, -\alpha_d)$ summability on T^d .

Theorem 6.3 (U. Goginava, A. Sahakian [16]). *Suppose $\alpha_1, \dots, \alpha_d \in (0, 1)$, $\alpha_1 + \dots + \alpha_d < 1/p$, $p \geq 1$. Then the class PBV_p is a class of $(C; -\alpha_1, \dots, -\alpha_d)$ summability on T^d .*

In [8] the first author has proved that the class PBV_p is not a class of $(C; -\alpha_1, \dots, -\alpha_d)$ summability on T^d , if $\alpha_1, \dots, \alpha_d \in (0, 1)$, and $\alpha_1 + \dots + \alpha_d \geq 1/p$.

Corollary 6.2 (U. Goginava, A. Sahakian [16]). *Suppose $\alpha_1, \dots, \alpha_d \in (0, 1)$, $\alpha_1 + \dots + \alpha_d < 1$. Then the set of functions f satisfying*

$$v_i(2^j, f) = O(2^{j\gamma}) \quad \text{for } i = 1, \dots, d,$$

is a class of $(C; -\alpha_1, \dots, -\alpha_d)$ summability on T^d .

REFERENCES

- [1] Bakhvalov, A. N. Continuity in Λ -variation of functions of several variables and the convergence of multiple Fourier series (Russian). *Mat. Sb.* 193, 12(2002), 3–20; English transl. in *Sb. Math.* 193(2002), No. 11–12, 1731–1748.
- [2] Bakhvalov, A. N. Continuity in Λ -variation and the summation of multiple Fourier series by Cesàro methods. (Russian) *Mat. Zametki* 90(2011), No. 4, 483–500; translation in *Math. Notes* 90(2011), No. 3–4, 469–484.
- [3] Bakhvalov, A. N. On the localization of Cesàro means of Fourier series of functions of bounded Λ -variation. (Russian) *Izv. Vyssh. Uchebn. Zaved. Mat.* 55(2011), no. 8, 9–13; translation in *Russian Math. (Iz. VUZ)* 55 (2011), No. 8, 7–10.
- [4] Chanturia, Z. A. The modulus of variation of a function and its application in the theory of Fourier series, *Soviet. Math. Dokl.* 15(1974), 67–71.
- [5] Dyachenko M. I, Waterman classes and spherical partial sums of double Fourier series, *Anal. Math.* 21(1995), 3–21.

- [6] Dyachenko M. I., Two-dimensional Waterman classes and u -convergence of Fourier series (Russian). *Mat. Sb.* 190(1999), No. 7, 23–40; English transl. in *Sb. Math.* 190(1999), No. 7-8, 955–972.
- [7] Dyachenko M. I.; Waterman D. , Convergence of double Fourier series and W-classes, *Trans. Amer. Math. Soc.* 357(2005), 397–407.
- [8] Goginava U. On the uniform summability of two-dimensional trigonometric Fourier series. *Proc. A. Razmadze Math. Inst.* 124 (2000), 55–72.
- [9] Goginava U., Uniform convergence of Cesáro means of negative order of double Walsh-Fourier series. *J. Approx. Theory.* 124(2003), 96–108.
- [10] Goginava U., On the uniform convergence of multiple trigonometric Fourier series. *East J. Approx.* 5(1999), No. 3, 253–266.
- [11] Goginava U., Sahakian A., On the convergence of Fourier series of functions of bounded partial generalized variation, *East J. Approx.* 16(2010), No. 2, 153–165.
- [12] Goginava U., Cesaro means of negative order of double Fourier series and Generalized Bounded variation, *Siberian Mathematical Journal*, 54(2013), , No. 6, pp. 1005–1013.
- [13] Goginava U., Sahakian A., On the convergence of Cesáro means of negative order of double trigonometric Fourier series of functions of bounded partial generalized variation, *Acta. Sci. Math. (Szeged)* 77(2011), 451–471.
- [14] Goginava U., Sahakian A., Convergence of double Fourier series and generalized Λ -variation, *Georgian Math. J.* 19(2012), No. 3, 497–509.
- [15] Goginava U., Sahakian A., On the convergence of multiple Fourier series of functions of bounded partial generalized variation, *Anal. Math.* 39(2013), No. 1, 45–56.
- [16] Goginava U., Sahakian A., On the summability of multiple Fourier series of functions of bounded partial generalized variation *Proc. Steklov Inst. Math.* 280(2013), 150–161.
- [17] Goginava U., Sahakian A., On the Convergence and Summability of double Walsh-Fourier series of functions of bounded generalized variation. *Journal of Contemporary Mathematical Analysis* 48(2014), No. 6.
- [18] Goginava U., Sahakian A., Convergence of Multiple Fourier Series of Functions of Bounded Generalized Variation (Submitted)
- [19] Hardy G. H., On double Fourier series and especially which represent the double zeta function with real and incommensurable parameters. *Quart. J. Math. Oxford Ser.* 37(1906), 53–79.
- [20] Jordan C., Sur la series de Fourier. *C.R. Acad. Sci. Paris.* 92(1881), 228–230.
- [21] Marcinkiewicz J. On a class of functions and their Fourier series. *Compt. Rend. Soc. Sci. Warsaw*, 26(1934), 71–77.
- [22] Sablin A. I., Λ -variation and Fourier series (Russian), *Izv. Vyssh. Uchebn. Zaved. Mat.* 10(1987), 66–68; English transl. in *Soviet Math. (Iz. VUZ)* 31(1987).
- [23] Sargsyan O. G., On the convergence and the Gibbs phenomenon of multiple Fourier series for functions of bounded harmonic variation (Russian). *Izv. Akad. Nauk Armenii Mat.*, 28(1993), No. 3, 3–20, English transl. in *J. Contemp. Math. Anal.*, 28(1993), No. 3.
- [24] Sahakian A. A., On the convergence of double Fourier series of functions of bounded harmonic variation (Russian). *Izv. Akad. Nauk Armyan. SSR Ser. Mat.* 21(1986), No. 6, 517–529; English transl., *Soviet J. Contemp. Math. Anal.* 21(1986), No. 6, 1–13.
- [25] Waterman D., On the summability of Fourier series of functions of Λ -bounded variation, *Studia Math.* 55(1976), 87–95.
- [26] Waterman D., On convergence of Fourier series of functions of generalized bounded variation. *Studia Math.*, 44(1972), No. 1, 107–117.
- [27] Wiener N., The quadratic variation of a function and its Fourier coefficients. *Massachusetts J. Math.*, 3(1924), 72–94.
- [28] Zhizhiashvili L. V., *Trigonometric Fourier series and their conjugates*, Kluwer Academic Publishers, Dordrecht, Boston, London, 1996.

[29] Zygmund A., Trigonometric series. Cambridge University Press, Cambridge, 1959.

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